International Journal of Theoretical Physics, Vol. 34, No. 9, 1995

Compact Linearly Ordered Effect Algebras

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Received January 18, 1995

Compact linearly ordered effect algebras are shown to be Archimedean MV-algebras with unique maximal ideal. Consequently they are isomorphic to the unit interval [0, 1] equipped with restricted addition.

1. INTRODUCTION

An *effect algebra* (Giuntini and Greuling, 1989; Foulis and Bennett, 1994) is a set E with two distinct particular elements $\mathbf{0}$ and $\mathbf{1}$ and with a partial binary operation $\oplus$: $E \times E \to E$ such that for all $a, b, c \in E$ we have:

(EA1) If $a \oplus b \in E$, then $b \oplus a \in E$ and $a \oplus b = b \oplus a$ (commutativity).
(EA2) If $b \oplus c \in E$ and $a \oplus (b \oplus c) \in E$, then $a \oplus b \in E$ and $(a \oplus b) \oplus c \in E$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).
(EA3) For each $a \in E$ there is a unique $b \in E$ such that $a \oplus b = \mathbf{1}$ (orthocomplementation).
(EA4) If $\mathbf{1} \oplus a$ is defined, then $a = \mathbf{0}$ (zero–one law).

If the assumptions of (EA2) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in E. Note that if (EA4) is changed to

(OA) If $a \oplus a$ is defined, then $a = \mathbf{0}$ (consistency)

we say that E is an *orthoalgebra*. Further note that *D-posets* introduced by Kôpka and Chovanec (1994) coincide with effect algebras in the sense that the difference in D-posets is connected with the sum in effect algebras via $a \oplus b = c$ if and only if $c \ominus a = b$. Hence any result obtained for effect algebras can be reformulated in the terms of D-posets and vice versa.

In several papers (Giuntini, n.d.; Chovanec, n.d.; Chovanec and Kôpka, n.d.; Mesiar, 1994), the connection between *MV-algebras* (multivalued alge-

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0020-7748/95/0900-1891$07.50/0

bras generalizing the Boolean algebras) and effect algebras (D-posets) were studied. Recall that an MV-algebra M was introduced by Chang (1957) in order to provide an algebraic proof of the completeness theorem of infinite-valued Lukasiewicz logic as a set with two distinct special elements **0** and **1** equipped with a binary operation $\oplus$: M $\times$ M $\to$ M and a unary operation *: M $\to$ M such that for all elements a, b, $c \in$ M we have:

(MV1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(MV2) $a \oplus \mathbf{0} = a$.
(MV3) $a \oplus b = b \oplus a$.
(MV4) $a \oplus \mathbf{1} = \mathbf{1}$.
(MV5) $(a^*)^* = a$.
(MV6) $\mathbf{0}^* = \mathbf{1}$.
(MV7) $a \oplus a^* = \mathbf{1}$.
(MV8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$ (Lukasiewicz axiom).

Given an MV-algebra M, we can define (after Mangani, 1973) three other binary operations on M, namely

$$a \odot b := (a^* \oplus b^*)^*$$

$$a \vee b := (a \odot b^*) \oplus b$$

$$a \wedge b := (a \oplus b^*) \odot b$$

Further, we can define a partial order on M by putting $a \leq b$ if and only if $a = a \wedge b$ (or equivalently if and only if $b = a \vee b$). The above lattice structure on an MV-algebra is allowed to be introduced due to the Lukasiewicz axiom (MV8).

2. EFFECT ALGEBRAS

Let (E, $\oplus$, **0**, **1**) be an effect algebra. We can extend the partial binary operation $\oplus$ on E to a (full) binary operation +: E $\times$ E $\to$ E by putting

$$\begin{aligned} a + b &= a \oplus b \quad \text{if} \quad a \oplus b \quad \text{is defined} \\ &= \mathbf{1} \quad \text{otherwise} \end{aligned}$$

Similarly we can introduce a unary operation *: E $\to$ E by putting $a^* = b$, where b is the unique element from (EA3) such that $a \oplus b = \mathbf{1}$. Further, we can introduce a partial order $\leq$ on E by putting

$$a \leq b \quad \text{if and only if} \quad a \oplus b^* \quad \text{is defined}$$

Examining the axioms (MV1)–(MV8) for MV-algebras for the structure (E, +, *, **0**, **1**), we can show that (MV1)–(MV7) are fulfilled. For details see

Giuntini (n.d.) or Chovanec (n.d.). Hence the axiom (MV8) is responsible for whether or not an effect algebra corresponds to an MV-algebra. Several sufficient and necessary conditions for (MV8) being fulfilled can be found in Chovanec (n.d.). Recall, e.g., that if E is a D-lattice, then (MV8) is equivalent with

$$(a - b) - c = (a - c) - b \qquad \text{for all} \quad a, b, c \in \mathrm{E}$$

or in the case when E is a D-poset with

$$b - (b - a) = a - (a - b) \qquad \text{for all} \quad a, b \in \mathrm{E}$$

where

$$\begin{aligned} a - b &= a \ominus b \qquad \text{if} \quad a \ominus b \quad \text{is defined} \\ &= \mathbf{0} \qquad \text{otherwise} \end{aligned}$$

We introduce another sufficient condition.

Proposition 1. Let (E, $\oplus$, **0**, **1**) be a totally ordered effect algebra. Then (E, +, *, **0**, **1**) is an MV-algebra.

Proof. Take arbitrary $a, b \in \mathrm{E}$. Then either $a \leq b$ or $b \leq a$. Suppose, e.g., that $a \leq b$. Then either $a = b$ and (MV8) is obviously fulfilled or $a < b$ and hence $a^* \oplus b$ is not defined. Consequently, $a^* + b = \mathbf{1}$ and thus

$$(a^* + b)^* + b = \mathbf{1}^* + b = \mathbf{0} + b = b \tag{1}$$

Further, $a < b$ implies that $c = a \oplus b^*$ exists in E and $c = a + b^* > \mathbf{0}$. Then

$$\mathbf{1} = c \oplus c^* = a \oplus b^* \oplus c^* = (a \oplus c^*) \oplus b^* = b \oplus b^*$$

and due to (EA3) we have that $b = a \oplus c^* = a + c^*$. It follows that

$$(b^* + a)^* + a = c^* + a = a + c^* = b \tag{2}$$

Equations (1) and (2) ensure (MV8). ∎

Remark 1. Let (E, +, *, **0**, **1**) be an MV-algebra arising from an effect algebra (E, $\oplus$, **0**, **1**). Then, for two elements $a, b \in \mathrm{E}$, $a \oplus b$ is defined if and only if either $a + b < \mathbf{1}$ (and then $a \oplus b = a + b$) or $a + b = \mathbf{1}$ and $b = a^*$ (and then $a \oplus b = \mathbf{1}$).

3. BOLD MV-ALGEBRAS

In the framework of MV-algebras, a special role is played by the *bold MV-algebras* of functions (defined on some nonempty universe X) with domain in the unit interval [0, 1]. In the case of a bold MV-algebra $\mathcal{M} \subset$ [0,

$1]^X$, $\mathbf{0}$ and $\mathbf{1}$ correspond to the minimal zero function and to the maximal unit function, respectively. Further, for f, $g \in \mathcal{M}$

$$f \oplus g = \min(f + g, \mathbf{1})$$
$$f^* = \mathbf{1} - f$$

where + and − are the usual arithmetic operations on real functions. Consequently, $f \odot g = \max(\mathbf{0}, f + g - \mathbf{1})$, the partial order $\leq$ is the usual partial order of real functions, and the binary operations $\vee$ and $\wedge$ are the common sup and inf of real functions.

Let M be an MV-algebra. If, for all elements $a, b \in$ M,

$$\text{n}.a = \underbrace{a \oplus \cdots \oplus a}_{\text{n times}} \leq b \qquad \text{for all n} \in \mathbf{N} \Rightarrow a^* \oplus b^* = a^* \tag{3}$$

then M is called an *Archimedean MV-algebra*.

Belluce (1986) discovered the structure of Archimedean MV-algebras.

Theorem 1 (Belluce, 1986). M is an Archimedean MV-algebra if and only if A is isomorphic with some bold MV-algebra $\mathcal{M}$.

Note that the domain X of the functions in the bold MV-algebra $\mathcal{M}$ in Theorem 1 is the set of all maximal ideals in M. For more details see Belluce (1986).

Proposition 2. Let $\mathcal{A}$ be a totally ordered Archimedean MV-algebra. Then and only then there is a strictly increasing mapping ϕ: $\mathcal{A} \to [0, 1]$, $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$, so that $\phi(\mathcal{A}) \subset [0, 1]$ is a bold MV-algebra and for all $a, b \in \mathcal{A}$ one has $a \oplus b = \phi^{-1}(\min(\phi(a) + \phi(b), 1))$ and $a^* = \phi^{-1}(1 - \phi(a))$.

The above proposition is a corollary of Theorem 1, taking into account that in a totally ordered set $\mathcal{A}$ there is a unique maximal ideal, namely $\mathcal{A}$, and hence the bold MV-algebra $\mathcal{M}$ from Theorem 1 is defined on a singleton. The mapping ϕ is simply the isomorphism from Theorem 1.

Now, we are ready to prove our main result.

4. TOTALLY ORDERED COMPACT EFFECT ALGEBRAS

Theorem 2. Let (E, $\oplus$, $\mathbf{0}$, $\mathbf{1}$) be a totally ordered compact (in the interval topology) effect algebra. Then there is a strictly increasing mapping ϕ: E $\to$ [0, 1] so that for all elements $a, b \in$ E, $a \oplus b$ is defined if and only if $\phi(a) + \phi(b) \leq 1$ and then

$$a \oplus b = \phi^{-1}(\phi(a) + \phi(b))$$

Proof. Due to Propositions 1 and 2, for the existence of a strictly increasing mapping ϕ fulfilling the requirements of this theorem, it is enough to prove the Archimedean property (3) with respect to the binary operation $+$ (the extension of $\oplus$).

Let $a, b \in$ E and let n.$a \leq b$ for all n $\in$ **N**. The sequence {n.a} is nondecreasing. Due to the compactness of E, $c = \sup$ n.a exists in E. Evidently $c \leq b$.

Suppose that {n.a} is a strictly increasing sequence. Then $a > \mathbf{0}$. From n.$a < c$ it follows that there is (the unique element) $e_n > \mathbf{0}$ so that

$$\mathrm{n}.a \oplus e_{\mathrm{n}} = \mathrm{n}.a + e_{\mathrm{n}} = c, \qquad \mathrm{n} \in \mathbf{N}$$

Put $e = \inf e_n$ (the existence of e follows again from the compactness of E). Then there is a unique element $u \in$ E such that

$$c = e \oplus u = e + u$$

It is evident that $u \leq c$ and that $u \geq$ n.a for all n $\in$ **N**. Consequently, $u \geq c$, which implies $u = c$ and thus $e = \mathbf{0}$. From the total order on E it follows that for some m $\in$ **N** one has $e_m < a$. Then

$$c = \mathrm{m}.a + e_{\mathrm{m}} < \mathrm{m}.a + a = (m + 1).a < c$$

a contradiction.

We have just shown that n.$a = c$ for all n $\geq$ m, where m is some element of **N**. Then $c = c + a = c +$ n.$a = c + c$ (by induction and due to the compactness of E), i.e., c is an idempotent element of (E, $+$). Then either $c \oplus c$ is not defined and consequently $c = c + c = \mathbf{1}$ or $c \oplus c$ is defined and then $\mathbf{1} = c \oplus c^* = (c \oplus c) \oplus c^* = c \oplus (c \oplus c^*) = c \oplus \mathbf{1}$ and by (EA3), $c = \mathbf{0}$.

If $c = \mathbf{0}$, then $a = \mathbf{0}$ and for all $b \in E$ one has

$$a^* + b^* = \mathbf{0}^* + b^* = \mathbf{1} + b^* = \mathbf{1} = a^*$$

If $c = \mathbf{1}$, then $b = \mathbf{1}$ and

$$a^* + b^* = a^* + \mathbf{1}^* = a^* + \mathbf{0} = a^*$$

what proves the Archimedean property (3).

By Remark 1 and Theorem 1 we see that $a \oplus b$ is defined only if $\phi(a) + \phi(b)$ is defined in [0, 1] and then (due to the isomorphism of MV-algebras) one has $a \oplus b = \phi^{-1}(\phi(a) + \phi(b))$. ■

Corollary 2. (E, $\oplus$, **0**, **1**) is a totally ordered, connected (i.e., for each $x < y$ from E there is $z \in$ E so that $x < z < y$), compact effect algebra if and only if it is isomorphic with the effect algebra ([0, 1], $\oplus$, 0, 1), where for $u, v \in [0, 1]$, $u \oplus v$ is defined iff $u + v \leq 1$ and then $u \oplus v = u + v$.

Proof. It is enough to show the "if" part only (and then we have to show the continuity of ϕ only), the opposite being obvious. The compactness of E ensures the continuity of ϕ from above in the minimal element **0**. The increasingness of ϕ ensures, for each nonextremal $a \in$ E, the existence of both the left and right limits of $\phi(x)$ in a. Take an arbitrary increasing sequence $\{x_n\} \subset$ E such that $\lim x_n = a$ and $\lim \phi(x_n) = d < \phi(a) < 1$. For each n $\in \mathbf{N}$ there is unique element e_n such that $x_n \oplus e_n = a$. Similarly, we see that the sequence $\{e_n\}$ strictly decreases to the minimal element **0**. Then the sequence $\{\phi(e_n)\}$ strictly decreases to 0 and $\phi(x_n) + \phi(e_n) = \phi(a)$. It follows that

$$\phi(x_n) = \phi(a) - \phi(e_n) \rightarrow \phi(a)$$

i.e., the mapping ϕ is left-continuous in a. Similarly we can show the left continuity in the maximal element **1** and the right continuity in an arbitrary nonextremal element $a \in$ E, which proves the continuity of ϕ. ■

Note that the mapping ϕ generating the operation $\oplus$ (and $+$) is called a *generator.* As an easy consequence of above theorem we have the characterization of effect algebras on closed subintervals of the extended real line (with the usual order).

Corollary 2. Let E $=$ [a, b] $\subseteq [-\infty, \infty]$ be a closed subinterval of the extended real line. Then (E, $\oplus$, a, b) is an effect algebra with the usual order of reals if and only if there is (unique!) increasing bijection g: [a, b] $\rightarrow$ [0, 1], which is called a generator of $\oplus$, so that for two elements x and $y \in$ E, $x \oplus y$ is defined whenever $g(x) + g(y) \leq 1$ and then

$$x \oplus y = g^{-1}(g(x) + g(y))$$

Remark 2. Note that a direct proof (without use of MV-algebras and Belluce's result) of Corollary 2 in the case E $=$ [0, 1] working with D-posets is given in Mesiar (n.d.; see also Mesiar, 1994). The generalization of the approach of Mesiar (n.d.) to arbitrary interval [a, b] is obvious.

Remark 3. In Theorem 2, neither the total ordering nor the compactness can be omitted. Take, e.g., E $= \{\mathbf{0}, a, b, \mathbf{1}\}$ so that $a = a^*$ and $b = b^*$. Then E is a compact effect algebra (and only $a \oplus b$, $a \oplus \mathbf{1}$, $b \oplus \mathbf{1}$, and $\mathbf{1} \oplus \mathbf{1}$ are not defined), but it is not totally ordered. Evidently, no generator ϕ: E $\rightarrow$ [0, 1] exists.

Following Mesiar and Pap (1994), let E $= \{a_n, b_n, n = 0, 1, 2, \ldots\}$, so that $a_0 = 0 < a_1 < a_2 < \cdots < c < \cdots < b_2 < b_1 < b_0 = 1$. Put

$$a_n \oplus a_m = a_{n+m} \qquad \text{for all} \quad m, n$$

$$a_n \oplus b_m = b_{m-n} \qquad \text{if} \quad m \geq n$$

otherwise $\oplus$ is not defined. Then (E, $\oplus$, 0, 1) is a totally ordered effect algebra, but it is not compact. Further, there is no generator ϕ generating $\oplus$. To see this, it is enough to recall that

$$n.a_1 = a_n$$

and if a generator ϕ would exist, then one should have $\phi(a_n) = n\phi(a_1)$. Since $\phi(a_1)$ should be positive and $\phi(a_n) < \phi(1) = 1$ for all $n \in \mathbf{N}$, we would get a contradiction.

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